24.2

5. Problem Use any of the theorems of Section 24.2.1 to determine the convergence or divergence of the following complex series.

$$\text{(a)} \sum_{n=1}^{\infty} \frac{n}{(2+i)^n} \qquad \text{(d)} \sum_{n=0}^{\infty} \left(\frac{1+n}{2+n}\right)^3 \qquad \text{(g)} \sum_{n=1}^{\infty} e^{-in}$$

Solution
(a) Use the ratio test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1}{2+i}(1+\frac{1}{n})\right| \tag{1}$$

$$\to \left|\frac{1}{2+i}\right| < 1 \quad as \quad n \to \infty. \tag{2}$$

Convergent.

(d) As $a_n \to 1$ when n tends to ∞, the series diverges. Note that if $\sum a_n$ converges, $a_n \to 0$ so its equivalent statement is that if a_n does not go to 0, $\sum a_n$ does not converge.

(g) Note that $|a_n| = |e^{-in}| = |\cos n + i \sin n| = 1$. Diverges.

6. Problem Use any of the theorem of Section 24.2.1 to determine, insofar as possible, the regions of convergence and divergence of the following power series.

$$\text{(a)} \sum_{n=0}^{\infty} z^{2n} \qquad \text{(d)} \sum_{n=4}^{\infty} e^n (z+i)^n \qquad \text{(g)} \sum_{n=0}^{\infty} e^{in} z^n$$

Solution
(a)

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{z^{2(n+1)}}{z^{2n}}\right| \tag{3}$$

$$= \left|z^2\right| < 1. \tag{4}$$

The region of convergence is $|z| < 1$.

(d)

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{e^{n+1}(z+i)^{n+1}}{e^n (z+i)^n}\right| \tag{5}$$

$$= e\,|z+i| < 1. \tag{6}$$

The region of convergence is $|z+i| < 1/e$, i.e. a disk centered at $z = -i$ with a radius of 1/e.

(g)

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{e^{i(n+1)}z^{n+1}}{e^{in}z^n}\right| && (7) \\ &= |z| < 1. && (8) \end{aligned}$$

The region of convergence is $|z| < 1$.

8. Problem Is the following power series the Taylor series of some function? If so, in what region does the series represent the function: if not, why not?

$$\text{(a)} \sum_{n=1}^{\infty} (-1)^{n+1} n (z-1)^n$$

Solution
(a)

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(-1)^{n+2}(n+1)(z-1)^{n+1}}{(-1)^{n+1}n(z-1)^n}\right| && (9) \\ &= \frac{n+1}{n}|z-1| && (10) \\ &= |z-1| < 1. && (11) \end{aligned}$$

Yes.
9. Problem Is the following power series the Taylor series of some function? If so, in what region does the series represent the function: if not, why not?

$$\text{(a)} \sum_{n=1}^{\infty} (-1)^{n+1} n (z-1)^n$$

Solution
The singular points of $f(z)$ are $z = 1$ and $z = 2$.
(a) $z = 1$ is the nearest singular point from $z = 0$ (center of expansion) so the radius of convergence is 1.
(b) $z = 1$ is the nearest singular point from $z = 3i$ so the radius of convergence is $|3i - 1| = \sqrt{10}$.

11. Problem Obtain the Taylor series of the given function, about $z = a$, and give its radius of convergence R.

(a) $\sin\, z,\ a=0$ (b) $\sin\, z,\ a=2-i$
(d) $e^{z^6},\ a=0$

Solution
(a)

$$\sin\, z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \tag{12}$$

convergent for $|z| < \infty$.
(b)

$$\begin{aligned}
\sin\, z &= \sin\left((z-a)+a\right) &(13)\\
&= \sin\left(z-a\right)\cos\, a + \cos\left(z-a\right)\sin\, a &(14)\\
&= \cos\, a\left((z-a) - \frac{(z-a)^3}{3!} + \frac{(z-a)^5}{5!} - \dots\right) + \sin\, a\left(1 - \frac{(z-a)^2}{2!} + \frac{(z-a)^4}{4!} + \dots\right) &(15)
\end{aligned}$$

where $a = 2-i$. Converges for $|z-a| < \infty$.
(d)

$$\begin{aligned}
e^{z^6} &= 1 + z^6 + \frac{(z^6)^2}{2!} + \frac{(z^6)^3}{3!} + \dots &(16)\\
&= 1 + z^6 + \frac{z^{12}}{2!} + \frac{z^{18}}{3!} + \dots &(17)
\end{aligned}$$

Converges for $|z| < \infty$.
12. Problem (*Binomial series*) (a) Derive the **binomial series**

$$\begin{aligned}
\frac{1}{(1-z)^m} &= 1 + mz + \frac{m(m+1)}{2!}\, z^2 + \frac{m(m+1)(m+2)}{3!}\, z^3 + \cdots\\
&= \sum_{n=0}^{\infty} \frac{(m+n-1)!}{(m-1)!n!}\, z^n \qquad (|z| < 1)
\end{aligned} \tag{12.1}$$

for any positive integer m.
(b) Use (12.1) to obtain the Taylor series of $1/(3-z)^2$ about $z = i$. Identify its disk of convergence. **HINT**: Write

$$\frac{1}{(3-z)^2} = \frac{1}{\left[3-(z-i)-i\right]^2} = \frac{1}{(3-i)^2\left[1-\left(\frac{z-i}{3-i}\right)\right]^2} \tag{12.2}$$

and use $(z-i)/(3-i)$ in place of z in (12.1).

Solution
(a) The formula of Taylor series around $z = 0$ is

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + \dots \tag{12.3}$$

where

$$\begin{aligned}
f(z) &= (1-z)^{-m} && (12.4)\\
f'(z) &= (-m)(1-z)^{-m-1}(-1) && (12.5)\\
f''(z) &= (-m)(-m-1)(1-z)^{-m-2}(-1)^2 && (12.6)\\
f'''(z) &= (-m)(-m-1)(-m-2)(1-z)^{-m-3}(-1)^3 && (12.7)\\
&= \dots && (12.8)\\
f^{(n)}(z) &= \underbrace{(-m)(-m-1)\dots(-m-(n-1))}_{n \text{ terms}}(1-z)^{-m-n}(-1)^n && (12.9)\\
&= (-1)^n(m+(n-1))\dots(m+1)m(1-z)^{-(m+n)}(-1)^n && (12.10)\\
&= \frac{(m+(n-1))!}{(m-1)!}(1-z)^{-(m+n)} && (12.11)
\end{aligned}$$

so

$$f^{(n)}(0) = \frac{(m+(n-1))!}{(m-1)!} \quad (12.12)$$

Therefore,

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n && (12.13)\\
&= \sum_{n=0}^{\infty} \frac{(m+n-1)!}{(m-1)!n!} z^n && (12.14)
\end{aligned}$$

(b) Follow the hint.

13. Problem Use (12.1) in Exercise 12 to obtain the Taylor expansions

$$\text{(a)}\ \frac{1}{(2z+1)^3} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+1)(n+2)2^n z^n \quad \text{in } |z| < \frac{1}{2}$$

Solution
(a) Let $z \to -2z$ and $m \to 3$ in (12.1).

$$\begin{aligned}
\frac{1}{(1+2z)^3} &= \sum_{n=1}^{\infty} \frac{(2+n)!}{2!n!}(-2z)^n && (12.15)\\
&= \frac{1}{2}\sum_{n=0}^{\infty}(n+2)(n+1)(-2)^n z^n && (12.16)\\
&= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+1)(n+2)2^n z^n && (12.17)
\end{aligned}$$

14. Problem Let $\sqrt{2}$ be defined by a principal-value branch cut (i.e., with the origin and negative x axis deleted and $-\pi < \theta < \pi$). Work out the first several terms of the Taylor series, about the given point, and give the region of validity of the Taylor series representation-that is, the region in which the series converges to the given function.

(a) $a = 1$

Solution

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots$$

and

$$\begin{aligned} f(z) &= \sqrt{z} && (12.18)\\ f'(z) &= \frac{1}{2\sqrt{z}} && (12.19)\\ f''(z) &= -\frac{1}{4}\frac{1}{z^{3/2}} && (12.20)\\ &\ldots \end{aligned}$$

(a) With the choice of the branch cut, $a = 1 = e^{0\pi i}$ so

$$\begin{aligned} f(a) &= (e^{0\pi i})^{1/2} = 1 && (12.21)\\ f'(a) &= \frac{1}{2\sqrt{1}} = \frac{1}{2} && (12.22)\\ f''(a) &= -\frac{1}{4} && (12.23) \end{aligned}$$

so

$$f(z) = 1 + \frac{1}{2}(z-1) - \frac{1}{8}(z-1)^2 + \ldots \qquad (12.24)$$

15. Problem Determine the coefficients of the next two terms in (40) (i.e., the z^4 and z^5 terms).
Solution
Use long division (easier said than done).

$$\frac{e^z}{\cos z} = 1 + z + z^2 + \frac{2z^3}{3} + \frac{z^4}{2} + \frac{3z^5}{10} + \ldots \qquad (12.25)$$

24.3

1. Problem
Derive the right-hand side of (26), up to and including the t^6 terms, by the method of undetermined coefficients.

Solution
Let

$$\frac{t}{\sin\,t} = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + \ldots \tag{12.26}$$

i.e.

$$\begin{aligned} t &= \sin\,t(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + \ldots) && (12.27)\\ &= (t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots)(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + \ldots) && (12.28) \end{aligned}$$

- 1st order: $a_0 = 1$.
- 2nd order: $a_1 = 0$.
- 3rd order: $a_2 - a_0/3! = 0$.
- 4th order: $a_3 - a_1/3! = 0$.
- 5th order: $a_4 - a_2/3! + a_0/5! = 0$.
- 6th order: $a_5 - a_3/3! + a_1/5! = 0$.
- 7th order: $a_6 - a_4/3! + a_2/5! - a_0/7! = 0$.

Solving for the unknowns above yields

$$\begin{aligned} a_0 &= 1 && (12.29)\\ a_1 &= 0 && (12.30)\\ a_2 &= \frac{1}{6} && (12.31)\\ a_3 &= 0 && (12.32)\\ a_4 &= \frac{7}{360} && (12.33)\\ a_5 &= 0 && (12.34)\\ a_6 &= \frac{31}{15120} && (12.35) \end{aligned}$$

2. Problem
Expand the function $f(x) = 1/[z(z-2)]$, in Example 3, about $z = i$, in the annulus between the two singular points, and show that

$$f(x) = -\frac{1}{2}\sum_{n=0}^{\infty}(-i)^n(z-i)^{-n-1} - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{2+i}{5}\right)^{n+1}(z-i)^n$$

in that annulus.

Solution

Note that $1/z$ needs to be expanded by Laurent series while $1/(z-2)$ should be expanded by Taylor series.

$$\begin{aligned}
\frac{1}{z} &= \frac{1}{(z-i)+i} && (12.36)\\
&= \frac{1}{(z-i)(1+\frac{i}{z-i})} && (12.37)\\
&= \frac{1}{z-i}\sum_{n=0}^{\infty}(\frac{-i}{z-i})^n && (12.38)\\
&= \sum_{n=0}^{\infty}(-i)^n(z-i)^{-n-1} && (12.39)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{z-2} &= \frac{1}{(z-i)-(2-i)} && (12.40)\\
&= \frac{1}{-(2-i)}\frac{1}{(1-\frac{z-i}{2-i})} && (12.41)\\
&= -\frac{1}{2-i}\sum_{n=0}^{\infty}(\frac{z-i}{2-i})^n && (12.42)\\
&= -\sum_{n=0}^{\infty}\frac{1}{(2-i)^{n+1}}(z-i)^n && (12.43)
\end{aligned}$$

4. Problem

Obtain the first three non-vanishing terms of the Laurent expansion of each of the following.

$$\text{(a) } \frac{1}{z} \quad \text{in } 1<|z-i|<\infty \text{(d) } \frac{1}{e^z-1} \quad \text{in } 0<|z|<2\pi \text{(h) } \frac{1}{z^2} \quad \text{in } 1<|z+i|<\infty$$

Solution

(a)

$$\begin{aligned}
\frac{1}{z} &= \frac{1}{(z-i)(1+\frac{i}{z-i})} && (12.44)\\
&= \frac{1}{z-i}(1-(\frac{i}{z-i})+(\frac{i}{z-i})^2-(\frac{i}{z-i})^3+\ldots) && (12.45)
\end{aligned}$$

(d) As

$$\begin{aligned}
e^z-1 &= z+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots && (12.46)\\
&\sim z && (12.47)
\end{aligned}$$

in the neighborhood of $z = 1$, it follows

$$\begin{aligned} \frac{1}{e^z - 1} &= \frac{z}{e^z - 1}\frac{1}{z} && (12.48) \\ &= f(z)\frac{1}{z} && (12.49) \end{aligned}$$

where

$$\begin{aligned} f(z) &= \frac{z}{e^z - 1} && (12.50) \\ &= f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \ldots && (12.51) \\ &= 1 - \frac{z}{2} + \frac{z^2}{12} + \ldots && (12.52) \end{aligned}$$

so

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \frac{z}{12} + \ldots \tag{12.53}$$

(h)

$$\begin{aligned} \frac{1}{z} &= \frac{1}{(z+i) - i} && (12.54) \\ &= \frac{1}{(z+i)(1 - \frac{i}{z+i})} && (12.55) \\ &= \frac{1}{z+i}\left(1 + \frac{i}{z+i} + (\frac{i}{z+i})^2 + \ldots\right) && (12.56) \end{aligned}$$

so

$$\begin{aligned} \frac{1}{z^2} &= \frac{1}{(z+i)^2}\left(1 + \frac{i}{z+i} + (\frac{i}{z+i})^2 + \ldots\right)^2 && (12.57) \\ &= \frac{1}{(z+i)^2}\left(1 + \frac{2i}{z+i} - \frac{3}{(z+i)^2} - \frac{4i}{(z+i)^3} + \ldots\right) && (12.58) \end{aligned}$$

(j) This is a little tough (and messy).
Note that

$$\tan z = \frac{\sin z}{\cos z}$$

is singular at $z = -\pi/2$ in the region concerned. In this neighborhood,

$$\begin{aligned} \cos z &= \cos\left((z + \frac{\pi}{2}) - \frac{\pi}{2}\right) && (12.59) \\ &= \cos\left(z + \frac{\pi}{2}\right)\cos\frac{\pi}{2} + \sin\left(z + \frac{\pi}{2}\right)\sin\frac{\pi}{2} && (12.60) \\ &= \sin\left(z + \frac{\pi}{2}\right) && (12.61) \\ &\sim z + \frac{\pi}{2} && (12.62) \end{aligned}$$

as $z \to -\pi/2$.
so

$$\tan z = \frac{1}{z+\pi/2}\frac{(z+\pi/2)\sin z}{\cos z} \tag{12.63}$$
$$= \frac{1}{z+\pi/2}g(z) \tag{12.64}$$

where
$$g(z) \equiv \frac{(z+\pi/2)\sin z}{\cos z} \tag{12.65}$$
which is regular in the region and therefore allows Taylor expansion as

$$g(z) = g(-\pi/2) + g'(-\pi/2)(z+\pi/2) + \frac{g''(-\pi/2)}{2!}(z+\pi/2)^2 + \dots \tag{12.66}$$
$$= -1 + \frac{1}{3}(z+\pi/2)^2 + \frac{1}{45}(z+\pi/2)^4 + \dots \tag{12.67}$$

so
$$\tan z = -\frac{1}{(z+\pi/2)} + \frac{1}{3}(z+\pi/2) + \frac{1}{45}(z+\pi/2)^3 + \dots \tag{12.68}$$

5 Problem
Determine all possible Taylor and Laurent expansions about the given point $z = a$, and state their regions of validity.

$$\text{(a)}\ \sin\frac{1}{z},\ a=0 \qquad \text{(b)}\ \frac{1}{z},\ a=-2 \qquad \text{(f)}\ \frac{\sin z}{z^4},\ a=0$$

Solution
(a)
$$\sin\frac{1}{z} = \frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} - \dots \tag{12.69}$$
converges for $|1/z| < \infty$, i.e. $|z| > 0$.

(b)

1. $|z+2| < 2$

$$\frac{1}{z} = \frac{1}{(z+2)-2} \tag{12.70}$$
$$= \frac{1}{-2(1-\frac{z+2}{2})} \tag{12.71}$$
$$= -\frac{1}{2}\left(1 + (\frac{z+2}{2}) + (\frac{z+2}{2})^2 + (\frac{z+2}{2})^3 + \dots\right) \tag{12.72}$$

2. $|z+2| > 2$

$$\frac{1}{z} = \frac{1}{(z+2)(1-\frac{2}{z+2})} \tag{12.73}$$

$$= \frac{1}{z+2}\left(1 + (\frac{2}{z+2}) + (\frac{2}{z+2})^2 + (\frac{2}{z+2})^3 + \ldots\right) \tag{12.74}$$

(f)

$$\frac{\sin\, z}{z^4} = \frac{1 - z^3/3! + z^5/51 - \ldots}{z^4} \tag{12.75}$$

$$= \frac{1}{z^4} - \frac{1}{3!}\frac{1}{z} + \frac{1}{5!}z + \ldots \tag{12.76}$$

6. Problem
A certain function $f(z)$ is represented by the expansion

$$\frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots$$

in $1 < |z| < \infty$. Determine the value of $f(z)$ at $z = 2i$ and at $z = i/3$.

Solution

$$f(z) = \frac{1}{z^2}\left(1 + (\frac{1}{z}) + (\frac{1}{z})^2 + (\frac{1}{z})^3 + \ldots\right) \tag{12.77}$$

$$= \frac{1}{z^2}\frac{1}{1-1/z} \tag{12.78}$$

$$= \frac{1}{z(z-1)} \tag{12.79}$$

so

$$f(2i) = \frac{1}{2i(2i-1)} \tag{12.80}$$

$$f(i/3) = \frac{1}{\frac{i}{3}(\frac{i}{3}-1)} \tag{12.81}$$

7. Problem
A certain function $f(z)$ is represented by the expansion

$$\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \cdots$$

Solution

$$
\begin{aligned}
f(z) &= \frac{1}{z}\left(1-(\frac{1}{z})+(\frac{1}{z})^2-(\frac{1}{z})^3+\ldots\right) && (12.82)\\
&= \frac{1}{z}\frac{1}{1+1/z} && (12.83)\\
&= \frac{1}{z+1} && (12.84)
\end{aligned}
$$

so

$$
\begin{aligned}
f(2) &= \frac{1}{3} && (12.85)\\
f(1/3) &= \frac{1}{1+1/3} && (12.86)
\end{aligned}
$$

8. Problem
Let $f(z) = \log z$ be defined by a principal value branch cut. Can f be expand in a Laurent series about $z = 0$? Explain.
Solution
You cannot expand $\ln z$ by Laurent series around $z = 0$ as you cannot select a domain as in Figure 2 (p.1226) centered at $z = 0$.

24.4

2. Problem
Determine the location of the zero (if any) of order 1 or higher, and their order, for each given function.

$$\text{(a) } z^2 - z\text{(d) } z\cos^2 z$$

Solution
(a) $z = 0, 1.$

(d)

$$z = 0, \frac{\pi}{2} \pm n\pi$$

3. Problem
Determine all singular points, in the finite z plane, of the following functions. If isolated, classify them further as Nth-order poles or essential singularities.

$$\text{(a) } \frac{e^z-1}{z^3}\text{(g) } \frac{z}{\sin^3 z}\text{(j) } \sinh \frac{1}{z}$$

Solution
(a) Note that $e^z - 1 \sim z$ as $z \to 0$ so

$$\frac{e^z-1}{z^3} = \frac{e^z-1}{z}\frac{1}{z^2} \qquad (12.87)$$

i.e. $z = 0$ is a second order pole.

(g) At $z = 0$, read the function as

$$\left(\frac{z}{\sin z}\right)^3 \frac{1}{z^2}$$

so $z = 0$ is a second order pole.
$z = n\pi$ $(n = \pm 1, \pm 2, \pm 3 \ldots)$ are third order poles.

(j)

$$\sinh \frac{1}{z} = \frac{1}{2}\left(e^{1/z} - e^{-1/z}\right)$$

so $z = 0$ is an essential singular point.

5. Problem
A function $f(z)$ is represented in a certain annulus by the given Laurent series. Classify the singularity (if any) of f at $z = 0$.

$$\text{(a) } \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots$$

Solution
(a)

$$\begin{aligned} f(z) &= \frac{1}{z^2}\left(1 + (\frac{1}{z}) + (\frac{1}{z})^2 + (\frac{1}{z})^3 + \ldots\right) && (12.88) \\ &= \frac{1}{z^2}\frac{1}{1 - 1/z} && (12.89) \\ &= \frac{1}{z(z-1)} && (12.90) \end{aligned}$$

i.e. $z = 0$ is a first order pole.

24.5

1. Problem
Let C_1 be a closed rectangular contour, traversed counterclockwise, with vertices at $-1 - i, 3 - i, 3 + 3i, -1 + 3i$. Let C_2 be a closed triangular contour, traversed clockwise, with vertices at $-2, 2$, and $-2 + 3i$. Evaluate the given integral by means of the residue theorem.

(a) $\displaystyle\oint_{C_1} \frac{dz}{\sin 2z}$ (d) $\displaystyle\oint_{C_1} \left(\frac{z+1}{z-1}\right)^3 dz$

Solution
(a) The singular points of $1/\sin 2z$ can be found by solving $\sin 2z = 0$ from which $2z = 0, \pm\pi, \pm 2\pi, \ldots$ so $z = 0, \pm\pi/2, \ldots$. Only $z = 0$ and $z = \pi/2$ are inside C_1 and both are first order poles.

$$\begin{aligned}
\text{Residue}(f,0) &= \frac{1}{2\cos\, 2z}|_{z\to 0} && (12.91)\\
&= \frac{1}{2} && (12.92)
\end{aligned}$$

$$\begin{aligned}
\text{Residue}(f,\pi/2) &= \frac{1}{2\cos\, 2z}|_{z\to \pi/2} && (12.93)\\
&= -\frac{1}{2} && (12.94)
\end{aligned}$$

so

$$\begin{aligned}
\oint_{C_1} \frac{dz}{\sin\, 2z} &= 2\pi i(\text{Residue}(0) + \text{Residue}(\pi/2)) && (12.95)\\
&= 0 && (12.96)
\end{aligned}$$

(d) $z = 1$ is the only singular point inside C_1 so

$$\oint_{C_1} f(z)dz = 2\pi\text{Residue}(f,1) \qquad (12.97)$$

Instead of using the residue formula for a third order pole, one can try the following:

$$\begin{aligned}
\frac{(z+1)^3}{(z-1)^3} &= \frac{((z-1)+2)^3}{(z-1)^3} && (12.98)\\
&= \frac{(z-1)^3 + 6(z-1)^2 + 12(z-1) + 8}{(z-1)^3} && (12.99)\\
&= \ldots + \frac{6}{(z-1)} + \ldots && (12.100)
\end{aligned}$$

so

$$\text{Residue}(f,1) = 6 \qquad (12.101)$$

and

$$\oint_{C_1} = 12\pi i \qquad (12.102)$$

2. Problem Evaluate by means of the residue theorem.

(a)

$$\int_0^\infty \frac{dx}{x^4 + a^4}$$

HINT: The zeros of $z^4 + a^4 = 0$ are $ae^{\pi i/4}$, $ae^{3\pi i/4}$, $ae^{5\pi i/4}$, and $ae^{7\pi i/4}$. Denote them as z_1, z_2, z_3, z_4, respectively. To work out the residue of $1/(z^4 + a^4)$ at z_1, for instance, it will be easier to evaluate

$$\lim_{z \to z_1} \left[(z - z_1) \frac{1}{z^4 + a^4} \right]$$

by *l'Hospital's rule* than to cancel the $(z-z_1)$'s and evaluate $1/[(z_1-z_2)(z_1-z_3)(z_1-z_4)]$. That step immediately gives the residue as $1/(4z_1^3)$.

(c)

$$\int_0^\infty \frac{x^2}{x^4+1} dx.$$

(d)

$$\int_0^\infty \frac{dx}{(x^2+1)^2}$$

(g)

$$\int_0^\infty \frac{\cos\ x}{(x^2+1)^2}\ dx$$

Solution
(a)

$$\begin{aligned}
2I &= \int_{-\infty}^\infty \frac{1}{x^4+a^4} dx && (12.103)\\
&= 2\pi i (\mathrm{Res}(\frac{1}{z^4+a^4}, ae^{\pi/4i}) + \mathrm{Res}(\frac{1}{z^4+a^4}, ae^{3\pi/4i})) && (12.104)\\
&= 2\pi i (\frac{1}{4a^3 e^{3\pi i/4}} + \frac{1}{4a^3 e^{9\pi i/4}}) && (12.105)\\
&= \frac{2\pi i}{4a^3}(e^{-3\pi i/4} + e^{-\pi i/4}) && (12.106)\\
&= \frac{\pi i}{a^3}(-)\sqrt{2} i = \frac{\sqrt{2}\pi}{2a^3}, && (12.107)
\end{aligned}$$

so

$$I = \frac{\sqrt{2}\pi}{4a^3}.$$

(c)

$$\begin{aligned}
2I &= \int_{-\infty}^\infty \frac{x^2 dx}{x^4+1} && (12.108)\\
&= 2\pi i (\mathrm{Res}(f, e^{\pi/4i}) + \mathrm{Res}(f, e^{3\pi/4i})) && (12.109)\\
&= 2\pi i (\frac{e^{\pi/2i}}{4e^{3\pi i/4}} + \frac{e^{3\pi/2}i}{4e^{9\pi i/4}}) && (12.110)\\
&= \frac{2\pi i}{4}(e^{-\pi i/4} + e^{-3\pi i/4}) = \frac{\pi i}{2}(-\sqrt{2} i) = \frac{\sqrt{2}\pi}{2}. && (12.111)
\end{aligned}$$

(d)

$$
\begin{aligned}
2I &= \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} && (12.112)\\
&= 2\pi i \mathrm{Res}(\frac{1}{(z^2+1)^2}, i) && (12.113)\\
&= 2\pi i \left(\frac{-i}{4}\right) = \frac{\pi}{2} && (12.114)
\end{aligned}
$$

(g)

$$
\begin{aligned}
2I &= \int_{-\infty}^{\infty} \frac{\cos\, x}{(x^2+1)^2} dx && (12.115)\\
&= \Re \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)^2} dx && (12.116)\\
&= \Re \left(2\pi i \mathrm{Res}(\frac{e^{iz}}{(z^2+1)^2}; i) \right) && (12.117)\\
&= 2\pi i (-\frac{i}{2e}) && (12.118)\\
&= \frac{\pi}{e} && (12.119)
\end{aligned}
$$

3. **Problem** Evaluate the given integral by means of the residue theorem. HINT: The contour in the z plane must be a closed loop, one time around. Thus, the limites on the polar angle mnust be any 2π interval, such as 0 to 2π or $-\pi$ to π. Thus, in (a) for instance, use the evenness of $\sin^2\, x$ to first rewrite the integral as $1/2 \int_{-\pi}^{\pi} \sin^2\, x$.

(a)

$$\int_0^{\pi} \sin^2\, x dx$$

(d)

$$\int_{\pi/2}^{\pi} \cos^2\, x dx$$

(g)

$$\int_0^{\pi} \sin^6\, x dx$$

(j)

$$\int_0^{\pi} \frac{dt}{1+\cos^2\, t dt}$$

(l)

$$\int_0^{2\pi} \frac{d\theta}{2+\sin\, 2\theta}$$

Solution

(a) Note that

$$
\begin{aligned}
2I &= \int_0^{2\pi} \sin^2 x dx && (12.120)\\
&= \oint \left(\frac{1}{2i}(z-\frac{1}{z})\right)^2 \frac{dz}{iz} && (12.121)\\
&= \oint \frac{i}{4}\frac{(z^2-1)^2}{z^3} dz && (12.122)\\
&= 2\pi i \text{Res}(z=0) && (12.123)\\
&= 2\pi i(-i/2) && (12.124)\\
&= \pi. && (12.125)
\end{aligned}
$$

Therefore $I = \pi/2$.

(d)

$$
\begin{aligned}
\int_{\pi/2}^{\pi} \cos^2 x dx &= \frac{1}{4}\int_0^{2\pi} \cos^2 x dx && (12.126)\\
&= \frac{1}{4}\oint (\frac{1}{2}(z+\frac{1}{z}))^2 \frac{dz}{iz} && (12.127)\\
&= \frac{1}{4}\oint \frac{-i(1+z^2)^2}{4z^3} dz && (12.128)\\
&= \frac{1}{4} 2\pi i(-i)\text{Res}\left(\frac{(1+z^2)^2}{4z^3}; 0\right) && (12.129)\\
&= \frac{1}{4} 2\pi i(-i)\frac{1}{2} && (12.130)\\
&= \frac{\pi}{4}. && (12.131)
\end{aligned}
$$

(g) Note that

$$
\begin{aligned}
2I &= \int_0^{2\pi} \sin^6 x dx && (12.132)\\
&= \oint \left(\frac{1}{2i}(z-1/z)\right)^6 \frac{dz}{iz} && (12.133)\\
&= \oint \frac{i}{64}\frac{(z^2-1)^6}{z^7} && (12.134)\\
&= 2\pi i \text{Res}(z=0) && (12.135)\\
&= 2\pi i(-5i/16) && (12.136)\\
&= 5\pi/8. && (12.137)
\end{aligned}
$$

So $I = 5\pi/32$.

(Please finish the rest by yourself.)
4. Problem
Using teh residue theorem, show that

$$\int_0^\pi \frac{\cos\, t dt}{1-2a\cos\, t + a^2} = \frac{\pi a}{1-a^2} \qquad (-1 < a < 1)$$

Solution

$$\begin{aligned}
2I &= \int_0^{2\pi} \frac{\cos\, t}{1-2a\cos\, t+a^2} dt && (12.138)\\
&= \oint \frac{1/2(z+1/z)}{1-2a\frac{1}{2}(z+1/z)+a^2}\frac{dz}{iz} && (12.139)\\
&= \oint \frac{i}{2}\frac{z^2+1}{z(az-1)(z-a)} && (12.140)
\end{aligned}$$

where $z=0$ and $z=a$ are both first order poles and are inside $|z|=1$ (note $|a|<1$!) so

$$\begin{aligned}
2I &= 2\pi i\frac{i}{2}(\mathrm{Res}(\frac{z^2+1}{z(az-1)(z-a)},0)+\mathrm{Res}(\frac{z^2+1}{z(az-1)(z-a)},a) && (12.141)\\
&= 2\pi i\frac{i}{2}(a+\frac{1+a^2}{a^3-a}) && (12.142)\\
&= \frac{2a\pi}{a^2-1}. && (12.143)
\end{aligned}$$

6 Problem Evaluate each by the residue theorem.
(a)

$$\int_0^\infty \frac{x^{a-1}}{x+1}dx \qquad (0<a<1)$$

(d)

$$\int_0^\infty \frac{\ln\, x}{1+x^2}dx$$

Solution
Let

$$f(z) \equiv \frac{z^{a-1}}{1+z}$$

$$\begin{aligned}
\oint f(z)dz &= 2\pi i \mathrm{Res}(e^{\pi i}) && (12.144)\\
&= 2\pi i e^{\pi i(a-1)} && (12.145)
\end{aligned}$$

Note that we used $e^{\pi i}$ rather than -1 in computing the residue because of the branch cut (you have to go counterclockwise to increase the angle). Computation of the contour integral along each segment ends up with (refer to your class note)

$$(1 - e^{2\pi i(a-1)}) \int_0^\infty \frac{x^{a-1}}{x+1} dx$$

so by equating them, one gets

$$
\begin{aligned}
I &= \frac{2\pi i e^{\pi i(a-1)}}{1 - e^{2\pi i(a-1)}} && (12.146)\\
&= \frac{2\pi i}{e^{-\pi i(a-1)} - e^{\pi i(a-1)}} && (12.147)\\
&= \frac{2\pi i}{-2i \sin \pi(a-1)} && (12.148)\\
&= \frac{-\pi}{\sin \pi(a-1)} && (12.149)
\end{aligned}
$$

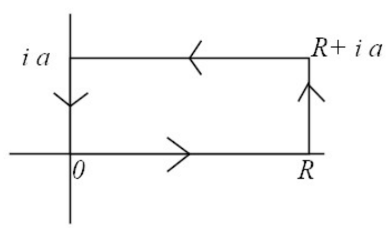
(d) This integral was discussed in class in detail. Please refer to your class note.

7. Problem
Evaluate

$$\int_0^\infty e^{-x^2} \cos 2ax dx = \frac{\sqrt{\pi}}{2} e^{-a^2}.$$

by integrating $\exp(-z^2)$ around a rectangle with vertices at 0, R, $R + ia$ and ia, and using the known integral

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$



Solution

1. Along Path I ($0 \to R$), set $z = x, dz = dx$ so that

$$
\begin{aligned}
\int_I &= \int_0^R e^{-x^2} dx && (12.150)\\
&\to \int_0^\infty e^{-x^2} dx \quad \text{as} \quad R \to \infty && (12.151)\\
&= \frac{\sqrt{\pi}}{2} && (12.152)
\end{aligned}
$$

2. Along Path II ($R \to R + ia$), set $z = R + iy, dz = idy$ and $y : 0 \to a$.

$$\int_{II} = \int_0^a e^{-(R+iy)^2} i dy \tag{12.153}$$
$$\to 0 \quad \text{as} \quad R \to \infty \tag{12.154}$$

because the order of the real part of the integrand is e^{-R^2}.

3. Along Path III ($R + ia \to ia$), set $z = x + ia$, $dz = dx$ and $x : R \to 0$.

$$\int_{III} = \int_R^0 e^{-(x+ia)^2} dx \tag{12.155}$$
$$= -\int_0^R e^{-x^2-2iax+a^2} dx \tag{12.156}$$
$$= -e^{a^2} \int_0^R e^{-x^2} (\cos 2ax - i \sin 2ax) dx \tag{12.157}$$
$$\to -e^{a^2} \left(\int_0^\infty e^{-x^2} \cos 2ax dx - i \int_0^\infty e^{-x^2} \sin 2ax dx \right) \tag{12.158}$$

4. Along Path IV ($ia \to 0$), $z = iy, dz = idy$ and $y : a \to 0$.

$$\int_{IV} = \int_a^0 e^{-(iy)^2} i dy \tag{12.159}$$
$$= -i \int_0^a e^{y^2} dy \tag{12.160}$$

Adding I + II + III + IV yields

$$\int_{I+II+III+IV} = \int_0^\infty e^{-x^2} dx + 0 - e^{a^2} \int_0^\infty e^{-x^2} (\cos 2ax - i \sin 2ax) dx - i \int_0^a e^{y^2} dy \tag{12.161}$$
$$= \left(\int_0^\infty e^{-x^2} dx - e^{a^2} \int_0^\infty e^{-x^2} \cos 2ax dx \right) \tag{12.162}$$
$$+ \; i \left(e^{a^2} \int_0^\infty e^{-x^2} \sin 2ax dx - \int_0^a e^{y^2} dy \right) \tag{12.163}$$

This contour integral is 0 because of the residue theorem (there is no singularity on and inside the contour !). By setting both the real and imaginary parts to 0, the above yields

$$\int_0^\infty e^{-x^2} dx = e^{a^2} \int_0^\infty e^{-x^2} \cos 2ax dx \tag{12.164}$$
$$e^{a^2} \int_0^\infty e^{-x^2} \sin 2ax dx = \int_0^a e^{y^2} dy \tag{12.165}$$

or

$$e^{a^2}\int_0^\infty e^{-x^2}\cos 2ax dx = \int_0^\infty e^{-x^2}dx \quad (12.166)$$
$$= \frac{\sqrt{\pi}}{2} \quad (12.167)$$

The equality from the imaginary part does not yield any meaningful (useful) result.

8. Problem Evaluate

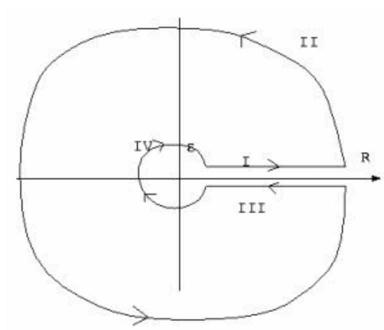
$$I \equiv \int_0^\infty \frac{dx}{x^2+x+1}$$

by considering

$$J \equiv \oint_C \frac{\log\ z}{z^2+z+1}dz$$

where $\log\ z$ is defined by the branch cut below and C is the contour shown in the same graph; let $\epsilon \to 0$ and $R \to \infty$. Thus show that

$$I = \frac{2\pi}{3\sqrt{3}}.$$



Solution
(a) Because the integrand is not an even function, the integration from 0 to ∞ is not equal to one half of the integral from $-\infty$ to ∞.

(b)

$$J = 2\pi i\left(\mathrm{Res}(\frac{\ln z}{z^2+z+1}; e^{2\pi i/3}) + \mathrm{Res}(\frac{\ln z}{z^2+z+1}; e^{4\pi i/3})\right) \quad (12.168)$$
$$= 2\pi i\left(\frac{2\pi i/3}{\sqrt{3}i} + \frac{4\pi i/3}{-\sqrt{3}i}\right) \quad (12.169)$$
$$= -\frac{2\pi}{3\sqrt{3}} \quad (12.170)$$

Note the angles we selected because of the branch cut. On the other hand, computation of the integral along the contour for each path ends up with two integrals (one from 0 to ∞ and the other from ∞ to 0 after one full rotation ($z = xe^{2\pi i}$) as

$$\int_0^\infty \frac{\ln x}{x^2+x+1}dx + \int_\infty^0 \frac{\ln(xe^{2\pi i})}{x^2+x+1}dx = \int_0^\infty \frac{\ln x}{x^2+x+1}dx - \int_0^\infty \frac{\ln\ x + 2\pi i}{x^2+x+1}dx \quad (12.171)$$

$$= -2\pi i \int_0^\infty \frac{dx}{x^2+x+1} \quad (12.172)$$

therefore, by equating the both, one obtains

$$I = \frac{2\pi}{3\sqrt{3}} \quad (12.173)$$

9 Problem Use the idea put forward in Exercise 8(b) to evaluate these integrals:
(a)

$$\int_0^\infty \frac{dx}{x^3+1}$$

Solution
(a) Consider

$$J \equiv \oint \frac{\ln\ z}{1+z^3}dz,$$

along the same contour as Prob. 8(b).
From the residue theorem, it follows

$$J = \oint \frac{\ln\ z}{1+z^3} \quad (12.174)$$

$$= 2\pi i(\mathrm{Res}(e^{\pi i/3}) + \mathrm{Res}(e^{\pi i}) + \mathrm{Res}(e^{5\pi i/3})) \quad (12.175)$$

$$= \ldots \quad (12.176)$$

$$= -\frac{4\pi^2}{3\sqrt{3}}i. \quad (12.177)$$

On the other hand, the contour integral of J yields

$$\int_0^\infty \frac{\ln\ x}{1+x^3}dx + \int_0^\infty \frac{\ln(xe^{2\pi i})}{1+x^3}dx = \int_0^\infty \frac{\ln\ x}{1+x^3}dx + \int_\infty^0 \frac{\ln x + 2\pi i}{1+x^3}dx \quad (12.178)$$

$$= -2\pi i \int_0^\infty \frac{dx}{1+x^3}, \quad (12.179)$$

so

$$\int_0^\infty \frac{dx}{1+x^3} = \frac{2\pi}{3\sqrt{3}}. \quad (12.180)$$

12. Problem The integral

$$I = \int_0^\infty \frac{\sin\ x}{x}dx = \frac{\pi}{2}$$

is well known, and there are serveral ways of evaluating it. Here, we ask you to evaluate it using the residue theorem.

Solution

Note that

$$I = \frac{1}{2}\Im \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx.$$

Note also that $z = 0$ is counted as one half of a regular pole so

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = \pi i \text{Residue}(0) = \pi i.$$