23.2

1. Problem Evaluate the following by expressing them in terms of real line integrals and then evaluating those integrals.

(a) $\int_C |z|^2 \, \mathrm{d}z$, where C is a straight line from $z = 0$ to $z = 1 + i$.

(d) $\int_C \mathrm{d}z/z$, where C consists of three straight-line segments: from $z = 1$ to $z = 1 - i$, from $z = 1 - i$ to $z = -1 - i$, and then from $z = -1 - i$ to $z = -1$

(f) $\int_C (Re\ z) \, \mathrm{d}z$, where C is a clockwise quarter circle from $z = 3i$ to $z = 3$ centered at $z = 0$.

(g) $\int_C (Im\ z) \, \mathrm{d}z$, where C is a straight line from $z = i$ to $z = 2 + 2i$.

Solution
(a) Note that this is a path-dependent integral.

$$\int |z|^2 dz = \int (x^2 + y^2)(dx + idy),$$

where $x = y$ and $dx = dy$ so

$$I = \int_0^1 2x^2(1 + i)dx = \frac{2(1 + i)}{3}.$$

(d) The integral path can be modified to the one along a semicircle of $|z| = 1$ from $\theta = 0$ to $\theta = -\pi$ (clockwise) as one can choose a closed loop that contains $z = 1$ and $z = -1$ but not $z = 0$ (singular point of $1/z$) so that $f(z) = 1/z$ is analytic on and inside such a loop. Thus,

$$\begin{aligned} \int \frac{1}{z} dz &= [\ln z]_1^{exp(-\pi i)} && (1) \\ &= \ln(e^{-\pi i}) - \ln 1 && (2) \\ &= -\pi i - 0 && (3) \\ &= \boxed{-\pi i}. && (4) \end{aligned}$$

Note that this integral is independent on the integral path.

(f) This integral is path-dependent as

$$f(z) = \Re z = (z + \bar{z})/2$$

is NOT analytic !. Along the path, one can set

$$\begin{aligned} z &= 3e^{i\theta} && (5) \\ dz &= 3ie^{i\theta} d\theta && (6) \\ x &= 3 \cos \theta && (7) \end{aligned}$$

so

$$\begin{aligned} \int_C x(dx + idy) &= \int_{\pi/2}^0 (3 \cos \theta)(3ie^{i\theta})d\theta && (8) \\ &= 9i \int_{\pi/2}^0 \cos \theta(\cos \theta + i \sin \theta)d\theta && (9) \\ &= -\frac{9\pi i}{4} + \frac{9}{2}. && (10) \end{aligned}$$

(g) This is a path dependent integral as $f(z) = y$ is not analytic. Along the path from $z = i$ to $z = 2 + 2i$ one can set

$$\begin{aligned} x &= 2t && (11) \\ y &= 1 + t \quad 0 < t < 1 && (12) \end{aligned}$$

so that

$$\begin{aligned} z &= x + iy && (13) \\ &= 2t + (1 + t)i && (14) \\ &= i + (2 + i)t && (15) \end{aligned}$$

and

$$dz = (2 + i)dt \tag{16}$$

so

$$\begin{aligned} \int_C y(dx + idy) &= \int_0^1 (1 + t)(2 + i)dt && (17) \\ &= \frac{3}{2}(2 + i). && (18) \end{aligned}$$

2. Consider

$$I = \int_C \bar{z}\,\mathrm{d}z,$$

where the initial and final points of C are $z = 0$ and $z = 1 + i$, respectively. Show that the integral is *path independent* by choosing two different paths and obtaining different values for I.

Solution

Since $\bar{z}$ is not analytic (see your class note for why), its integral is path-dependent.

23.3

1.Problem According to Example 2,

$$\int_C \frac{\mathrm{d}z}{z^2} = 0 \tag{19}$$

where C is a counterclockwise circle of radius R, centered at the origin. Yet $f(z) = 1/z^2$ is not analytic within C, it is singular at $z = 0$. Explain why this result does not violate Cauchy's theorem.

Solution

Cauchy's theorem is a sufficient but NOT necessary condition for

$$\oint_C f(z)dz = 0.$$

i.e. the contour integral being 0 does not necessary imply that the function, $f(z)$, is regular on and inside the contour although the converse is true (Cauchy's theorem !).

2.Problem Consider $I = \int_C \mathrm{d}z/z$, where C is the counterclockwise unit circle, and the assertion that $I = 0$, from Cauchy's theorem, because $f(z) = 1/z$ is analytic in the domain D containing C. (See the accompanying figure.) Yet we show in Example 2 that $I = 2\pi i$. Explain the apparent contradiction.

Solution

Note that the domain D is not singly-connected (there is a hole) so Cauchy's theorem does not apply.

4.Problem Let C_1, C_2, C_3 be the following simple closed curves:
$C_1 : |z| = 1$, counterclockwise
$C_2 : |z| = 1$, clockwise
$C_3 :$ thesquarewithverticesat$1 - i, 1 + i, -1 + i, -1 - i$, counterclockwise.

Evaluate each of the following integrals using Cauchy's theorem if applicable and partial fractions if necessary.

(a) $\int_{C_1} Re\ z \mathrm{d}z$

(d) $\int_{C_3} \frac{\mathrm{d}z}{z^2-3}$

(g) $\int_{C_2} \frac{\mathrm{d}z}{z(z+5)}$

(j) $\int_{C_3} \frac{\mathrm{d}z}{|z|}$

Solution
(a) Along C_1, one can set $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$, $0 < \theta < 2\pi$, $x = \cos\,\theta$ and $y = \sin\,\theta$. So

$$\begin{aligned} \int_0^{2\pi} \cos\,\theta i e^{i\theta} d\theta &= i\int_0^{2\pi} \cos\,\theta(\cos\,\theta + i \sin\,\theta) d\theta && (20) \\ &= \pi i. && (21) \end{aligned}$$

(d) Note that both of the singular points, $z = \sqrt{3}$ and $z = -\sqrt{3}$, are outside the contour, C_3 so $\int = 0$.

(g) Partial fraction of $1/(z^2+5z)$ is $(1/z-1/(z+5))/5$ so $z = 0$ is the only singular point inside C_2. Thereby,

$$\begin{aligned} \oint_{C_2} \frac{dz}{z(z+5)} &= \frac{1}{5}\oint_{C_2} \frac{dz}{z} - \frac{1}{5}\oint_{C_2} \frac{dz}{z+5} && (22) \\ &= -\frac{2\pi i}{5} - 0 && (23) \\ &= = \frac{2\pi i}{5}. && (24) \end{aligned}$$

(j) From $-1 - i$ to $1 - i$, $y = -1, dy = 0$ and x varies from -1 to 1 so

$$\int \frac{dz}{|z|} = \int_{-1}^{1} \frac{dx}{\sqrt{x^2+1}}.$$

From $1 - i$ to $1 + i$, $x = 1, dx = 0$ and y varies from -1 to 1 so

$$\int \frac{dz}{|z|} = \int_{-1}^{1} \frac{dy}{\sqrt{y^2+1}}.$$

From $1 + i$ to $-1 + i$, $y = 1, dy = 0$ and y varies from 1 to -1 so

$$\int \frac{dz}{|z|} = \int_{1}^{-1} \frac{dx}{\sqrt{x^2+1}}.$$

From $-1 + i$ to $-1 - i$, $x = -1, dx = 0$ and y varies from 1 to -1 so

$$\int \frac{dz}{|z|} = \int_{1}^{-1} \frac{dy}{\sqrt{y^2+1}}.$$

Adding the four integrals above amounts to $\boxed{0}$ (not because of Cauchy's theorem !!)

5.Problem Can we use path deformation to obtain

$$\int_{C_3} \bar{z}\mathrm{d}z = \int_{C_1} \bar{z}\mathrm{d}z, \tag{25}$$

where C_1 and C_3 are defined in Exercise 4 ? Explain.
Solution
The formula,

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz \tag{26}$$

is true if $f(z)$ has isolated singular points. $\bar{z}$ is singular everywhere except for $z = 0$. So this is NOT TRUE.

Here is a direct approach.
Along C_1:

$$x = \cos\,\theta, \quad y = \sin\,\theta$$

so

$$dx = -\sin\,\theta, \quad dy = \cos\,\theta$$

$$\oint (x - iy)(dx + idy) = \oint (\cos\,\theta - i\sin\,\theta)(-\sin\,\theta + i\cos\,\theta)d\theta = \int_0^{2\pi} id\theta = 2\pi i.$$

Along C_3:

1. From $z = -1 - i$ to $z = 1 - i$, $y = -1, \quad dy = 0$, x varies from -1 to 1. So

$$\int \bar{z}dz = \int_{-1}^{1} (x + i)dx = 2i.$$

2. From $z = 1 - i$ to $z = 1 + i$, $x = 1, \quad dx = 0$, y varies from -1 to 1. So

$$\int \bar{z}dz = \int_{-1}^{1} (1 - iy)idy = 2i.$$

3. From $z = 1 + i$ to $z = -1 + i$, $y = 1, \quad dy = 0$, x varies from 1 to -1. so

$$\int \bar{z}dz = \int_{1}^{-1} (x - i)idx = 2i.$$

4. From $z = -1 + i$ to $z = -1 - i$, $x = -1, \quad dx = 0$, y varies from 1 to -1. so

$$\int \bar{z}dz = \int_{1}^{-1} (-1 - iy)idy = 2i.$$

So

$$\oint_{C_3} \bar{z}dz = 8i.$$

6.Problem Evaluate $\int_C z^{20}\mathrm{d}z$, where C is the path

(a) $y = x - x^3$ from $x = 0$ to $x = 1$

Solution
(a) As z^{20} is analytic, its integration is independent of the path. Thereby, take the straight line from (0,0) to (1,0) instead of the given curve.

$$\int_C z^{20} dz = \int_0^1 x^{20} dx = \frac{1}{21}.$$

8. Problem *(Path deformation in multiply-connected domain)* Show that if $f(z)$ is analytic in the shaded region between and on the contours C, C_1, C_2 (see the accompanying figure), then

$$\int_C f(z)\,\mathrm{d}z = \int_{C_1} f(z)\,\mathrm{d}z + \int_{C_2} f(z)\,\mathrm{d}z. \tag{27}$$

You may use the result stated in Exercise 3. NOTE: More generally,

$$\int_C f(z)\,\mathrm{d}z = \int_{C_1} f(z)\,\mathrm{d}z + \cdots + \int_{C_n} f(z)\,\mathrm{d}z \tag{28}$$

if $C_1, \ldots, C_n$ are non-intersecting counterclockwise closed contours within C, and f is analytic between and on $C, C_1, \ldots, C_n$.
Solution
Shown in class.

9.Problem Evaluate the following integrals, where in each case C is the circle $|z| = 3$, counterclockwise.

(a) $\int_C \frac{\mathrm{d}z}{z(z-1)}$

(d) $\int_C \frac{z\mathrm{d}z}{z^2-3z+2}$

Solution
(a)

$$\oint \frac{dz}{z(z-1)} = \oint \frac{dz}{z-1} - \oint \frac{dz}{z} = 2\pi i - 2\pi i = 0.$$

(d)

$$\oint \frac{zdz}{(z-1)(z-2)} = -\oint \frac{dz}{z-1} + 2\oint \frac{dz}{z-2} = -2\pi i + 2(2\pi i) = 2\pi i.$$

23.4

3. Problem
Use the fundamental theorem to evaluate each of the following.
(a) $\int_0^i z\,dz$
(b) $\int_{-i}^i z^4\,dz$
(c) $\int_4^{-1-2i} (e^{-z} - 3z^2)\,dz$
(d) $\int_i^0 \cos 3z\,dz$
(e) $\int_{1-i}^{1+i} ze^z\,dz$
(f) $\int_4^{-i} z \sin z\,dz$
(g) $\int_0^{3i} ze^{z^2}\,dz$
(h) $\int_0^{1+2i} \sin^2 z\,dz$

(i) $\int_{-2i}^{3} z\cos 2z\, dz$
(j) $\int_{i}^{2+i} \cosh 3z\, dz$
(k) $\int_{0}^{i} \cos^3\ z\, dz$
(l) $\int_{i}^{1} z^2 e^{2z^3}\, dz$

Solution
(a)

$$\int_0^i z dz = \left[\frac{z^2}{2}\right]_0^i = -1/2.$$

(d)

$$\int_i^0 \cos\, 3z dz = \left[\frac{\sin\, 3z}{3}\right]_i^0 = -\frac{\sin\, 3i}{3}.$$

(g)

$$\int_0^{3i} ze^{z^2} dz = \left[\frac{e^{z^2}}{2}\right]_0^{3i} = \frac{1}{2e^9} - \frac{1}{2}.$$

4. Problem
Determine all possible values of

$$I = \int_{1-i}^{1+i} \frac{dz}{z(z-1)}.$$

Solution Note that

$$\ln A - \ln B = \ln \frac{A}{B} \tag{29}$$

is also true for complex numbers.

$$\begin{aligned}
I &= \int_{1-i}^{1+i} \frac{dz}{z-1} - \int_{1-i}^{1+i} \frac{dz}{z} &(30)\\
&= [\ln(z-1)]_{1-i}^{1+i} - [\ln z]_{1-i}^{1+i} &(31)\\
&= \ln i - \ln(-i) - (\ln(1+i) + \ln(1-i)) &(32)\\
&= \ln\left(\frac{i}{-i}\right) - \ln\left(\frac{1+i}{1-i}\right) &(33)\\
&= \ln(-1) - \ln(i) &(34)\\
&= \ln\left(\frac{-1}{i}\right) &(35)\\
&= \ln(i) &(36)\\
&= \left(\frac{\pi}{2} + 2n\pi\right) i, \quad n = 0, \pm 1, \pm 2 \ldots &(37)
\end{aligned}$$

23.5

1. Problem Evaluate each integral, where C is the counterclockwise circle $|z| = 3$. Use Cauchy's integral formula or its extension.

$$\text{(a)} \oint_C \frac{\cos z}{z}\,dz \text{(d)} \oint_C \frac{z^2-1}{z^2+1}\,e^z\,dz \text{(g)} \oint_C \frac{\sinh 3z}{(z^2+1)^2}\,dz \text{(i)} \oint_C \frac{e^{z^2}}{z\cos(z/2)}\,dz \text{(j)} \oint_C \frac{z}{(z+i)(z^2+1)}\,dz$$

Solution
(a) $I = 2\pi i \cos z|_{z=0} = 2\pi i$.

(d) Note that

$$\frac{1}{z^2+1} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right) \tag{38}$$

so

$$\begin{aligned} I &= \oint \frac{1}{2i}\left(\frac{z^2-1}{z-i}e^z - \frac{z^2-1}{z+i}e^z\right)dz & (39)\\ &= \left[\pi i(z^2-1)e^z\right]_{z=i} - \left[\pi i(z^2-1)e^z\right]_{z=-i} & (40)\\ &= \pi i(i^2-1)e^i - \pi i((-i)^2-1)e^{-i} & (41)\\ &= -4\pi i \sin 1. & (42) \end{aligned}$$

(g)

$$\begin{aligned} I &= \oint_{C_1} \frac{\sinh 3z dz}{(z+i)^2(z-i)^2} + \oint_{C_2} \frac{\sinh 3z dz}{(z+i)^2(z-i)^2} & (43)\\ &= \frac{2\pi i}{1!}\frac{d}{dz}\frac{\sinh 3z}{(z+i)^2}|_{z=i} + \frac{2\pi i}{1!}\frac{d}{dz}\frac{\sinh 3z}{(z-i)^2}|_{z=-i} & (44)\\ &= 2\pi i(-\frac{3\cos 3}{4} + \frac{\sin 3}{4}) + 2\pi i(-\frac{3\cos 3}{4} + \frac{\sin 3}{4}) & (45)\\ &= \pi i(\sin 3 - 3\cos 3). & (46) \end{aligned}$$

where C_1 is a little circle around $z = i$ and C_2 is a little circle around $z = -i$.

(j) Note that

$$f(z) = \frac{z}{(z+i)^2(z-i)} \tag{47}$$

$$\begin{aligned} \oint_C \frac{z}{(z+i)^2(z-i)}dz &= \oint_{C_1} \frac{g(z)}{z-i}dz + \oint_{C_2} \frac{h(z)}{(z+i)^2}dz & (48)\\ &= 2\pi i g(i) + 2\pi i h'(i) & (49)\\ &= 2\pi i\frac{i}{(2i)^2} + 2\pi i\frac{-i}{(-2i)^2} & (50)\\ &= 0 & (51) \end{aligned}$$

where $g(z) = z/(z+i)^2$, $h(z) = z/(z-i)$, C_1 is a loop that encircles $z = i$ and C_2 is a loop that encircles $z = -i$.

2.
Problem (*Important little integral*) In Section 23.3 we show that

$$\oint_C (z-a)^n\,dz = \begin{cases} 2\pi i & (n=-1) \\ 0 & (n \neq -1), \end{cases} \tag{2.1}$$

where n is any integer and a is within the contour C. Derive (2.1) using Cauchy's theorem, the Cauchy integral formula, and the generalized Cauchy integral formula.

Solution

If $n \geq 0$, $I = 0$ by the Cauchy's theorem. If $n = -1$, $I = 2\pi i(1)|_{z=a} = 2\pi i$ by the Cauchy's integral formula. If $n < -1$,

$$I = \frac{2\pi i}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}(1)|_{z=a} = 0.$$

3. Problem (a) Show from (22) that if C is a circle of radius ρ with center at z, $f(z)$ is analytic inside and on C, and M is the maximum value of $|f(z)|$ on C, then

$$\left|f^{(n)}(z)\right| \leq \frac{n!M}{\rho^n}. \tag{3.1}$$

(b) (*Liouville's theorem*) Use (3.1) to prove **Liouville's theorem**: *If f is entire (i.e., analytic for all finite z) and bounded for all z, then f is a constant.*

(c) Since $f(z) = \sin z$ is entire and not a constant, it must not be bounded (according to Liouville's theorem). Demonstrate that, in fact, it is not bounded.

(d) (*Fundamental theorem of algebra)* Use Liouville's theorem to prove the **Fundamental theorem of algebra**: *if $P(z)$ is a polynomial function of z, of degree 1 or greater*,

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0 \qquad (a_n \neq 0)$$

then $P(z) = 0$ has at least one root. **HINT**: Suppose that $P(z)$ is nonzero everywhere. Then $f(z) = 1/P(z)$ is analytic everywhere and is bounded.

Solution

(a) According to eq.(22),

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint \frac{f(z)}{(z-a)^{n+1}}dz \tag{3.2}$$

so

$$\begin{aligned}
|f^{(n)}(a)| &\leq \frac{n!}{2\pi}\oint \frac{|f(z)|}{|z-a|^{n+1}}dz & (3.3)\\
&\leq \frac{n!}{2\pi}\frac{M}{\rho^{n+1}}\oint |dz| & (3.4)\\
&= \frac{n!}{2\pi}\frac{M}{\rho^{n+1}}2\pi\rho & (3.5)\\
&= \frac{n!M}{\rho} & (3.6)
\end{aligned}$$

(b) Let $n = 1$, then $|f'(z)| \leq \frac{M}{\rho}$. Because ρ can be arbitrarily large, it follows that $f'(z)$ can be arbitrarily small, *i.e.*, $f'(z) = 0$ for each z so that $f(z)$ is a constant.

(c) For example, $|\sin yi| \to \infty$ as $y \to \infty$.

(d) Will be shown in class.