21.2

5(b). (Problem) Show that

$$|z^n| = |z|^n \quad \text{and} \quad \left|\frac{1}{z^n}\right| = \frac{1}{|z|^n} \quad \text{for} \quad n = 1, 2, \ldots$$

(Solution) (b) Use polar form, i.e. let $z = re^{i\theta}$, then $|z^n| = r^n = |z|^n$. Note $|e^{i\theta}| = |\cos\theta + i\sin\theta| = 1$.

9.(Problem) Evaluate each of the following, that is, express each in standard Cartesian form $x + iy$.

(a)
$$(2-i)^3$$

(e)
$$\left(\frac{1+i}{2-i}\right)^3$$

(g)
$$\Im(1+i)^3$$

(h)
$$\left(\Re\frac{1}{1+i}\right)^3$$

(Solution)
(a) $(2-i)^3 = 2 - 11i$.
(e)
$$\left(\frac{1+i}{2-i}\right)^3 = -\frac{26}{125} - \frac{18}{125}i.$$
(g) Note $1 + i = \sqrt{2}e^{\pi i/4}$ and $(1+i)^3 = 2^{3/2}e^{3\pi i/4}$ so $\Im(1+i)^3 = 2^{3/2}\sin(3\pi/4) = 2$.
(h) $1/8$.

21.3

1.(Problem) With a labeled sketch, show the point sets defined by the following.

(a) $|z-1| \le 4$

(c) $|z+2-i| < 2$

(i) $|z+1| = |z| + 1$

(Solution)
(a) $|z-1| \le 4$ can be read as a set of points inside a circle with a radius of 4 centered at $z = 1$.
(c) Read it as $|z-(-2+i)| < 2$, *i.e.* a set of points inside the circle with a radius 2 centered at $-2+i$.
(i) Square the both sides to get $|z+1|^2 = |z|^2 + 2|z| + 1$ which is expanded to $(x+1)^2 + y^2 = x^2 + y^2 + 2\sqrt{x^2+y^2} + 1$ from which one gets $x = \sqrt{x^2+y^2}$, *i.e.* $y = 0$ and $x > 0$ (don't forget this condition !).

2.(Problem) Determine the range R for the given function. Include sketches of both the domain D and the range R, and give the equations of any curved parts of the boundary of R.

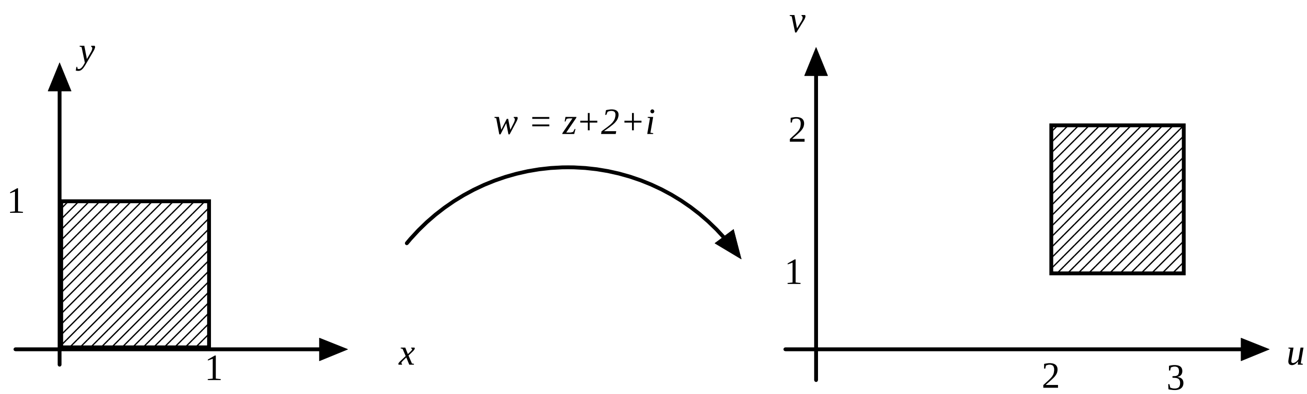
(a) $w(z) = z + 2 + i$ on $0 < x < 1$, $0 < y < 1$

(d) $w(z) = z^2$ on $-\infty < x < 0$, $0 < y < \infty$

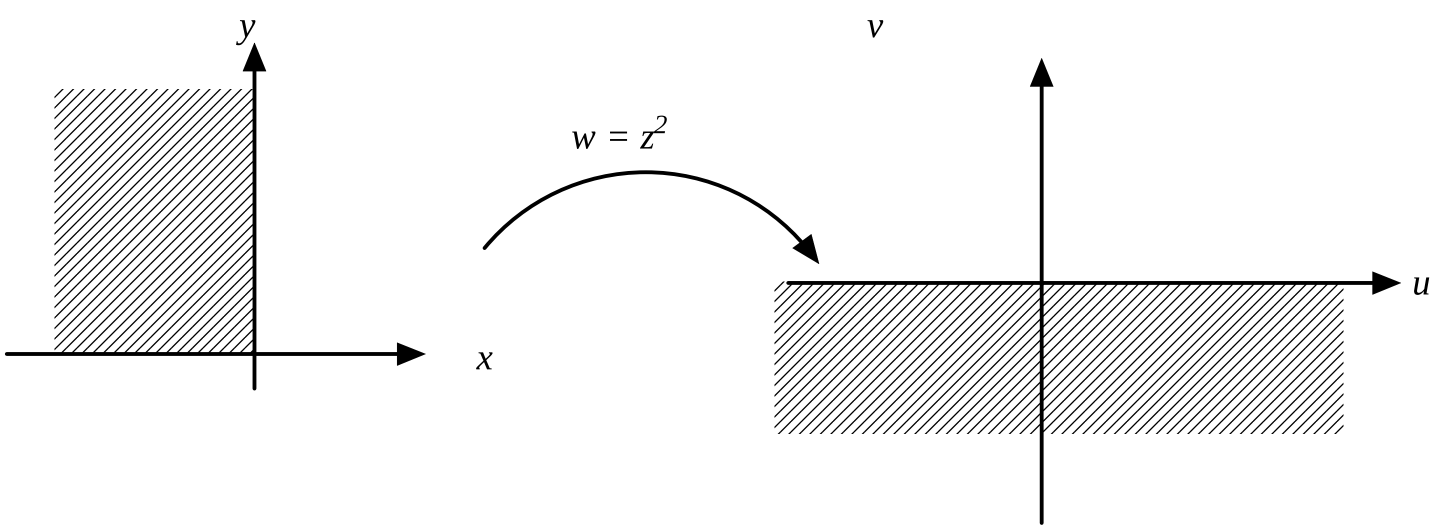
(f) $w(z) = iz^2$ on $0 < x < 1$, $0 < y < 1$

(g) $w(z) = z^3$ on $0 < x < \infty$, $0 < y < \infty$

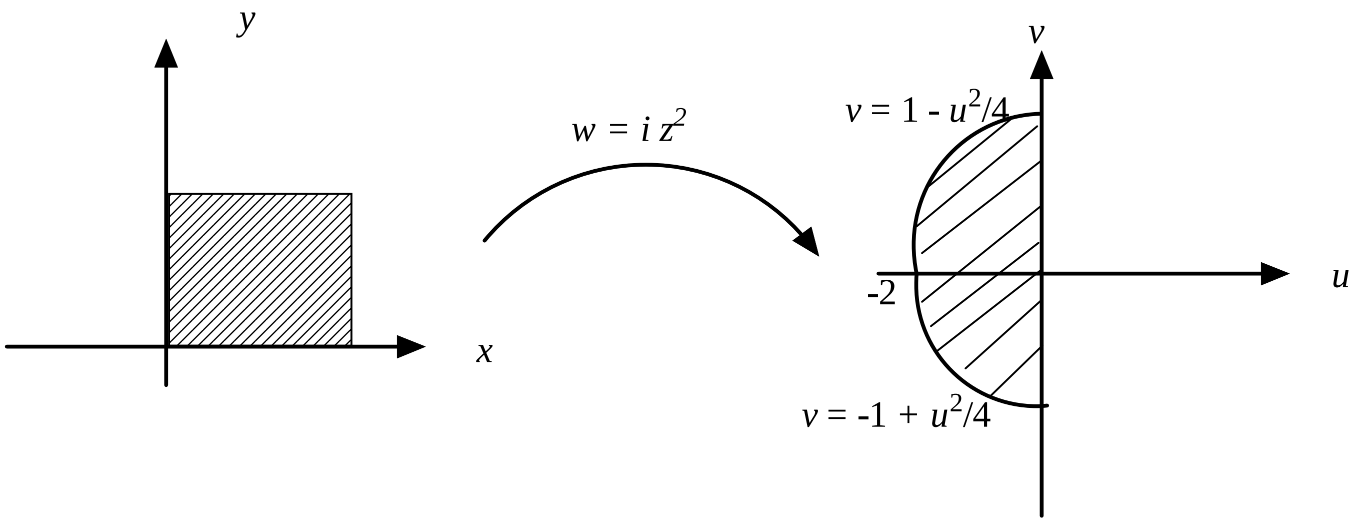
(Solution)
(a) Shift the rectangle $(0 < x < 1,\ 0 < y < 1)$ to the right by 2 and up by 1.



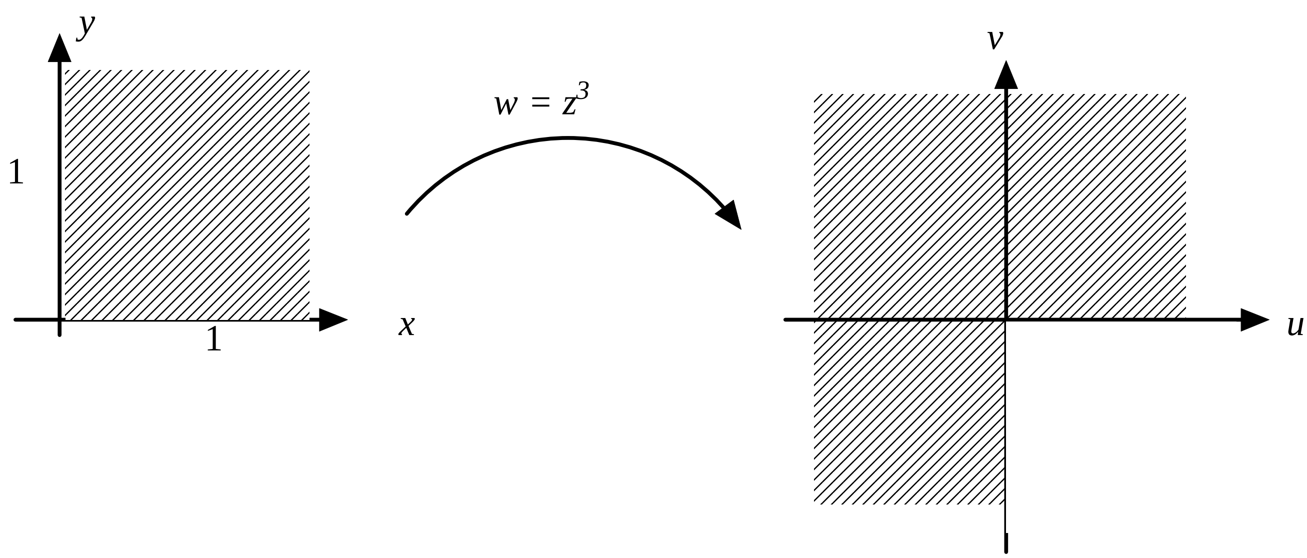
(d) Note that with $w = z^2$, the argument on z is doubled. So the positive y axis ($\theta = \pi/2$) is mapped to the negative v axis ($\phi = \pi$) and the negative x axis ($\theta = \pi$) is mapped to the positive u axis ($\phi = 2\pi$), i.e. the area expressed by $v \leq 0$.



(f) $w = iz^2$ is equivalent to $u + iv = i(x + iy)^2$. Comparing both the real part and the imaginary part gives $u = -2xy$ and $v = x^2 - y^2$. Using this relationship, one can transform the boundary of the rectangle $(0 < x < 1, 0 < y < 1)$. *i.e.* $\{\ x = 0, 0 < y < 1\} \to \{u = 0, -1 < v < 0\}$, $\{\ y = 0, 0 < x < 1\} \to \{u = 0, 0 < v < 1\}$, $\{\ x = 1, 0 < y < 1\} \to \{u = -2y, v = 1 - y^2\} \to \{v = 1 - u^2/4\}$, $\{y = 1, 0 < x < 1\} \to \{u = -2y, v = y^2 - 1\} \to \{v = u^2/4 - 1\}$.



(g) Note that with $w = z^3$, the argument is tripled. So the positive x axis ($\theta = 0$) remains the positive u axis while the positive y axis ($\theta = \pi/2$) is mapped to the negative v axis ($\theta = 3\pi/2$). So the first quadrant is tripled on the w plane.



3.(Problem) Show whether or not $|e^z| = e^{|z|}$. More generally, is $|w(z)| = w(|z|)$?
(Solution)

$$e^z = e^{(x+iy)} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

Thus,

$$|e^z| = e^x |\cos y + i \sin y| = e^x.$$

On the other hand,

$$e^{|z|} = e^{\sqrt{x^2+y^2}},$$

so

$$|e^z| \neq e^{|z|}.$$

Counterexample: $z = iy$. $|e^z| = |\cos y + i \sin y| = 1$ while $e^{|z|} = e^y \neq 1$. In general, $|w(z)| \neq w(|z|)$.

4.(Problem) Show whether or not

$$\bar{e^z} = e^{\bar{z}}.$$

More generally, is

$$w\bar{(z)} = w\left(\bar{z}\right)?$$

(Solution)
Yes as

$$\overline{e^z} = \overline{e^x(\cos\, y + i\sin\, y)} = e^x(\cos\, y - i\sin\, y),$$

and

$$e^{\bar{z}} = e^{x-iy} = e^x(\cos\, y - i\sin\, y).$$

This is not true in general. Counterexample: $w(z) = i$.

7.(Problem) Show that

(a) $\sin\,(-z) = -\sin\, z$ and $\cos\,(-z) = \cos\, z$

(b) $\cos\,(z_1 + z_2) = \cos\, z_1 \cos\, z_2 - \sin\, z_1 \sin\, z_2$

(e) $\cos\,(x + iy) = \cos\, x \cosh\, y - i\sin\, x \sinh\, y$

(f) $\sin\,(x + iy) = \sin\, x \cosh\, y + i\cos\, x \sinh\, y$

(g) $\cosh^2 z - \sinh^2 z = 1$

(Solution)
(a)(b) Both are proven by *analytical continuation*. If you are not satisfied by this, use the definition of equations (14) and (15).

(e)

$$\cos\left(x + iy\right) = \cos\, x \cos\, iy - \sin\, x \sin\, iy,$$

Note that

$$\cos\left(iy\right) = \frac{1}{2}\left(e^{i(iy)} + e^{-i(iy)}\right) = \frac{1}{2}(e^{-y} + e^{y}) = \cosh\, y,$$

and

$$\sin\left(iy\right) = \frac{1}{2i}\left(e^{i(iy)} - e^{-i(iy)}\right) = \frac{i}{2}\left(e^{y} - e^{-y}\right) = i\sinh\, y,$$

so

$$\cos\left(x + iy\right) = \cos\, x \cosh\, y - i\sin\, x \sinh\, y.$$

(f) $\sin\left(x + iy\right) = \sin\, x \cos\, iy + \cos\, x \sin\, iy = \sin\, x \cosh\, y + i\cos\, x \sinh\, y$.
(g)

$$\begin{aligned}
\cosh{}^2 z - \sinh{}^2 z &= \left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2 && (1)\\
&= \frac{e^{2z} + 2 + e^{-2z}}{4} - \frac{e^{2z} - 2 + e^{-2z}}{4} && (2)\\
&= 1. && (3)
\end{aligned}$$

8.(Problem)
Show that
(b)

$$\cosh\left(z_1 + z_2\right) = \cosh\, z_1 \cosh\, z_2 + \sinh\, z_1 \sinh\, z_2.$$

(g)

$$\cosh^2 z - \sinh^2 z = 1.$$

Solution
(b) Expand

$$\frac{e^{(z_1+z_2)} + e^{-(z_1+z_2)}}{2}$$

and

$$\frac{e^{z_1} + e^{-z_1}}{2}\frac{e^{z_2} + e^{-z_2}}{2} + \frac{e^{z_1} - e^{-z_1}}{2}\frac{e^{z_2} - e^{-z_2}}{2}$$

separately and verify that the both are the same.
(g)

$$(\frac{e^z + e^{-z}}{2})^2 - (\frac{e^z - e^{-z}}{2})^2 = \frac{e^{2z} + 2 + e^{-2z} - (e^{2z} - 2 + e^{2z})}{4} = 1.$$

9. (Problem) Evaluate each of the following in standard Cartesian form.

(a) $e^{2+\pi i}$

(d) $\sin(3 + \pi i)$

(Solution)
(a) $e^2 e^{\pi i} = e^2(\cos\pi + i\sin\pi) = -e^2$.

(d)

$$\begin{aligned}
\sin(3+\pi i) &= \frac{e^{i(3+\pi i)} - e^{-i(3+\pi i)}}{2i} \\
&= \frac{e^{3i}e^{-\pi} - e^{-3i}e^{\pi}}{2i} \\
&= \dots \\
&= \cosh\pi\ \sin 3 + i\cos 3\ \sinh\pi.
\end{aligned}$$

11. Prove that

(a) $e^z = 1$ if and only if $z = 2n\pi i$, where n is any integer.

(b) $e^{z_1} = e^{z_2}$ if and only if $z_1 = z_2 + 2n\pi i$, where n is any integer.

(Solution)
(a) This comes from $1 = e^{2n\pi i}$ (polar form).
(b) $e^{z_1} = e^{z_2}$ is equivalent to $e^{z_1 - z_2} = 1$. From the result of (a), $z_1 - z_2$ must be equal to $2n\pi$.
13.(Problem) Show that the range R of the function $\sin z$ on the semi-infinite strip $-\pi/2 < x < \pi/2$, $0 < y < \infty$ is shown in the accompanying figure.
(Solution)
$\sin z = \sin(x + iy) = \sin x\cosh y + i\cos x\sin hy$, i.e. $u = \sin x\cosh y$ and $v = \cos x\sin hy$. The bottom ($y = 0, -\pi/2 < x < \pi/2$) is mapped to $u = \sin x, v = 0$ or $-1 < u < 1, v = 0$. The right edge ($x = \pi/2, y > 0$) is mapped to $u = \cosh y, v = 0$ or $1 < u < \infty, v = 0$. The left edge ($x = -\pi/2, y > 0$) is mapped to $u = -\cosh y, v = 0$ or $-\infty < u < 1, v = 0$. Combining them, it is found that the rectangle is mapped to the upper half plane.

21.4

1.(Problem) Determine r and the principal argument θ_0 (in radians and in degrees) for each of the following values of z.

(a) $-3i$

(f) $2 - 12i$

(Solution)
(a)

$$-3i = 3e^{3/2\pi i}.$$

(f)

$$2 - 12i = 2\sqrt{37}e^{-1.405i}.$$

4.(Problem) Obtain z^{10} and z^{20}, in both polar and Cartesian form, for each given z.

(a) $-1 + i$

(Solution)
(a)

$$z = \sqrt{2}e^{3\pi/4i}.$$
$$z^{10} = 2^5 e^{7.5\pi i} = -32i.$$

$$z^{20} = 2^{10}e^{15\pi i} = -1024.$$

5.(Problem) Find all values of $z^{1/2}$ and $z^{1/5}$ for each given z. Express those values in polar form, and show their location in the z plane, as we have done in Fig. 5.

(a) i

(Solution)
(a) $z = e^{\pi/2i}$ so $z^{1/2} = e^{\pi/4i}$ and $z^{1/5} = e^{\pi/10i}$.

6.(Problem) Obtain, in Cartesian form, all values of $\log z$ for each given z.

(a) -2

(Solution)
(a) $-2 = 2e^{\pi i + 2n\pi i}$ so $\ln(-2) = \ln 2 + (2n+1)\pi i$.

8.(Problem) Obtain, in polar form, all values of $z^{2/3}$, $z^{3/2}$, and z^{π} for each given z.

(a) $2i$

(Solution)
(a) $z = 2e^{(\pi/2+2n\pi)i}$ so $z^{2/3} = 2^{2/3}e^{2/3(\pi/2+2n\pi)i} = 2^{2/3}e^{\pi/3i}e^{4n/3\pi i}$.

11.(Problem) Obtain, in Cartesian form, the principal values of $\log z$ and $\sqrt{z}$ for each given z.

(a) $-3i$

(Solution)
(a) $\ln(-3i) = \ln\left(3e^{(3\pi/2+2n\pi)i}\right) = \ln 3 + i(3\pi/2 + 2n\pi)$. By setting $n = 0$, the principal value of $\ln(-3)$ is $\ln 3 + 3\pi i/2$.
$\sqrt{-3i} = \sqrt{(3e^{(3\pi/2+2n\pi)i})} = \sqrt{3}e^{(3\pi/2+2n\pi)i/2}$. By setting $n = 0$, the principal value of $\sqrt{-3i}$ is

$$\sqrt{3}e^{3\pi i/4} = \sqrt{3}\left(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right).$$

13.(Problem) *(Inverse of sine function)* We define the inverse of the sine function

$$w(z) = \arcsin z \tag{4}$$

such that $z = \sin w$

(a) Writing the latter as
$$z = \left(e^{iw} - e^{-iw}\right)/2i, \tag{5}$$
show that $e^{iw} = iz + \left(1 - z^2\right)^{1/2}$, and hence that
$$\arcsin z = -i \log\left[iz + \sqrt{1 - z^2}\right]. \tag{6}$$

(b) Observe that arcsin z is multi-valued because of the $\left(1 - z^2\right)^{1/2}$ and also because of the log[ ]. Specifically, for each value of $z\,(\neq \pm 1)$, the $\left(1 - z^2\right)^{1/2}$ gives two values. Then, for each of these values the log gives an infinite set of values. To illustrate this point, show that
$$\arcsin\frac{1}{2} = \frac{\pi}{6} + 2k\pi \quad \text{or} \quad \frac{5\pi}{6} + 2k\pi \tag{7}$$
for $k = 0, \pm 1, \pm 2, \ldots$

(c) Determine all possible values of arcsin 2.

(d) Determine all possible values of arcsin $(2i)$.

(Solution)
(a) $z = (e^{iw} - e^{-iw})/(2i)$ is equivalent to $2iz = e^{iw} - e^{-iw}$ or $(e^{iw})^2 - 2ize^{iw} - 1 = 0$. Solving this quadratic equation, one gets $e^{iw} = iz + (1 - z^2)^{1/2}$, i.e. $w = \arcsin z = -i\ln(iz + (1 - z^2)^{1/2})$. Note that $(1 - z^2)^{1/2}$ is multi-valued ($\pm$) without an appropriate branch cut which should take care of the $\pm$ sign when solving the quadratic equation.

(b) Note that $z^{1/2} = (re^{\theta i + 2n\pi i^{1/2}}) = \sqrt{r}e^{\frac{\theta}{2}i + n\pi i}$ where the values of $e^{n\pi i}$ is either 1 or -1 depending on whether n is even or odd. Therefore, $(1 - (1/2)^2)^{1/2} = \pm\frac{\sqrt{3}}{2}$. $arcsin\,(1/2) = -i\ln(i/2 \pm \sqrt{3}/2) = -i\ln(e^{\pi/6i + 2k\pi i})\text{or} - i\ln(e^{5\pi/6i + 2k\pi i}) = \frac{\pi}{6} + 2k\pi\text{or}\frac{6\pi}{6} + 2k\pi$.

(c) $\arcsin 2 = -i\ln(2i + \sqrt{-3}) = -i\ln((2 \pm \sqrt{3})i) = -i\ln((2 \pm \sqrt{3})e^{(\pi/2 + 2k\pi)i}) = -i(\ln(2 \pm \sqrt{3}) + i(\pi/2 + 2k\pi)) = (\pi/2 + 2k\pi) - i\ln(2 \pm \sqrt{3})$.

(d) $\arcsin(2i) = -i\ln(i(2i) + (1 - (2i)^2)^{1/2}) = -i\ln(-2 \pm \sqrt{5}) = -2n\pi - i\ln(\sqrt{5} - 2)\text{or} - (2n+1)\pi - i\ln(\sqrt{5} + 2)$.

14. (Problem) *(Inverse of other trigonometric functions)* Proceeding as in Exercise 13, derive these formulas.

(a) $\arccos z = -i\log\left[z + \sqrt{z^2 - 1}\right]$

(b) $\arctan z = -\frac{i}{2}\log\frac{i - z}{i + z}$

(c) $\cot^{-1} z = -\frac{i}{2}\log\frac{z + 1}{z - 1}$

(Solution)
(a) Solve $z = \frac{e^{iw}+e^{-iw}}{2}$ for e^{iw} to get $arccos\, z = -i\ln(z + \sqrt{z^2-1}^{1/2})$.

(b) Solve $z = \frac{e^{iw}-e^{-iw}}{i(e^{iw}+e^{-iw})}$ to get $arctanz = -\frac{i}{2}\ln\frac{i-z}{i+z}$.

(c) Similar to (b).

21.5

10.(Problem) Given $f(z)$, use (19) to obtain $f'(z)$. Express your answer in terms of z.

(a) $\cos z$

(e) $\frac{1}{z}\,(z \neq 1)$

(Solution)
(a) $f'(z) = -\sin z$.
(e) $f'(z) = -1/z^2$.
If $f(z)$ is a regular function, you can differentiate it as if it were a real valued function.

11.(Problem) Given $f(z)$, determine $f'(z)$, where it exists, and state where f is analytic and where it is not.

(a)
$$\left(1-2z^3\right)^5$$

(b)
$$\frac{x+iy}{x^2+y^2}$$

(c)
$$|z|\sin z$$

(d)
$$\frac{1}{z^2+3iz-2}$$

(e)
$$\frac{1}{z^3+1}$$

(f)
$$x + i\sin y$$

(Solution)
(a) $f(z)$ is regular so $f'(z) = 5(1-2z^3)^4 \times (-6z^2)$.

(b) Note that
$$\frac{x+iy}{x^2+y^2} = \frac{z}{z\bar{z}} = \frac{1}{\bar{z}}$$
which is NOT analytic (contains $\bar{z}$).

(c) This is not analytic as it contains
$$|z| = \sqrt{z\bar{z}}.$$

(d)

$$f'(z) = \frac{-(2z+3i)}{(-2+3iz+z^2)^2}.$$

(e)

$$f'(z) = \frac{-3z^2}{(1+z^3)^2}.$$

(f) This is not an analytic function (use C-R to verify).

12.(Problem) *(Cauchy-Riemann equations in polar coordinates)* Derive the Cauchy-Riemann equations (30) in the manner indicated.

(a) By carrying out the limit in (8). HINT: First let $\Delta z \to 0$ along the constant-θ line through z_0, and then let $\Delta z \to 0$ along the constant-r line through z_0. Pay careful attention to your expression for Δz in each of these cases because these cases are trickier than the cases of a horizontal approach ($\Delta z = \Delta x$) and a vertical approach ($\Delta z = i\Delta y$), used in (16) and (17).

(b) By making the change of variables $x = r\cos\theta$, $y = r\sin\theta$ in (18).

(Solution)
(a) On the constant θ line, $\Delta z = \Delta r e^{i\theta}$ (r varies but θ remains constant) so

$$\begin{aligned}
f'(z) &= \lim_{\Delta r\to 0} \frac{u(r+\Delta r,\theta)+iv(r+\Delta r,\theta)-u(r,\theta)-iv(r,\theta)}{\Delta r e^{i\theta}} && (8)\\
&= \lim_{\Delta r\to 0} e^{-i\theta}\Big(\frac{u(r+\Delta r,\theta)-u(r)}{\Delta r}+i\frac{v(r+\Delta r,\theta)-v(r)}{\Delta r}\Big) && (9)\\
&= e^{-i\theta}\,(u_r+iv_r). && (10)
\end{aligned}$$

On the constant-r line, $\Delta z = i\ r\ \Delta\ \theta\ e^{i\theta}$ (r is constant while θ varies) so

$$\begin{aligned}
f'(z) &= \lim_{\Delta\theta\to 0} \frac{u(r,\theta+\Delta\theta)+iv(r,\theta+\Delta\theta)-u(r,\theta)-iv(r,\theta)}{i\ r\Delta\theta\ e^{i\theta}} && (11)\\
&= \frac{1}{r}\ e^{-i\theta}\ v_\theta - \frac{i}{r}\ e^{-i\theta}\ u_\theta. && (12)
\end{aligned}$$

By comparing the real and imaginary parts, one obtains

$$\begin{aligned}
u_r &= \tfrac{1}{r}v_\theta\\
v_r &= -\tfrac{1}{r}u_\theta.
\end{aligned} \qquad (13)$$

(b) Recall the chain differentiation rule, i.e.

$$\begin{aligned}
u_x &= u_r r_x + u_\theta \theta_x && (14)\\
v_y &= v_r r_y + v_\theta \theta_y, && (15)
\end{aligned}$$

and

$$
\begin{aligned}
r_x &= \frac{\partial r}{\partial x} \\
&= \frac{\partial}{\partial x}\sqrt{x^2+y^2} \\
&= \frac{x}{\sqrt{x^2+y^2}} \\
&= \frac{r\cos\theta}{r} \\
&= \cos\theta \\
& \quad \text{etc...}
\end{aligned}
$$

So $u_x = v_y \rightarrow u_r r_x + u_\theta \theta_x = v_r r_y + v_\theta \theta_x$ or

$$u_r \cos\theta - u_\theta \frac{\sin\theta}{r} = v_r \sin\theta + v_\theta \frac{\cos\theta}{r}.$$

and
$u_y = -v_x \rightarrow u_r r_y + u_\theta \theta_y = -v_r r_x - v_\theta \theta_y$
or

$$u_r \sin\theta + u_\theta \frac{\cos\theta}{r} = -v_r \cos\theta + v_\theta \frac{\sin\theta}{r}.$$

Solving the above simultaneous equations for u_r and v_r yields $u_r = \frac{1}{r}v_\theta$ and $v_r = -\frac{1}{r}u_\theta$.

13.(Problem) Determine where these functions are differentiable and where they are analytic, by checking for satisfaction of the Cauchy-Riemann equations and for continuity of u, v and their first-order partial derivatives.

(a) $f(z) = z^{100}$

(b) $f(z) = \sqrt{z}$, defined by the branch cut shown in Fig. 6

(c) $f(z) = \frac{1}{\sqrt{z}}$, where $\sqrt{z}$ is defined by the branch cut shown in Fig.6

(Solution)
(a) $f(z) = z^{100}$ is analytic as it is a function of z alone. $f'(z) = 100z^{99}$.

(b) $f(z) = \sqrt{z}$ is analytic except for $z = 0$ at which point $f'(z)$ fails to exist.

$$f'(z) = \frac{1}{2\sqrt{z}}.$$

(c) $f(z) = \frac{1}{\sqrt{z}}$ is analytic except for $z = 0$ at which point $f'(z)$ fails to exist.

$$f'(z) = -\frac{1}{2}z^{-3/2}.$$

14.(Problem) (a) $f = u + iv$ and $\bar{f} = u - iv$. The Cauchy-Riemann relation requires that $u_x = v_y$, $u_y = -v_x$, $u_x = -v_y$ and $u_y = v_x$ from which $u_x = u_y = 0$ so $u = \mathit{const}$ then, $v_x = v_y = 0$ so $v = \mathit{const}$ too.

(b) By the fundamental theorem of calculus.

15. Determine whether or not the given function u is harmonic and, if so, in what region. If it is, find the most general conjugate function v and corresponding analytic function $f(z)$. Express f in terms of z.

(a) $e^x \cos y$

(b) $e^{2x} \sin 2y$

(c) $x^3 - 3xy^2$

(d) $r^3 \sin 3\theta$

(e) $r^2 \cos 2\theta + 4$

(f) r

(g) $x \cos 2x \cosh 2y + y \sin 2x \sinh 2y$

(Solution)
(In the following solution, add a constant C to $f(z)$.)
(a) Yes. $v = e^x \sin y$ so that $f(z) = e^z$.

(b) Yes. $v = -e^{2x} \cos 2y$. Note that

$$\begin{aligned} u + iv &= e^{2x}(\sin 2y - i \cos 2y) \\ &= e^{2x}(-i^2 \sin 2y - i \cos 2y) \\ &= -ie^{2x}(\cos 2y + i \sin 2y) \\ &= -ie^{2x}e^{2yi} \\ &= -ie^{2(x+iy)} \\ &= -ie^{2z}. \end{aligned}$$

(c) Yes. $v = 3x^2y - y^3$ so $f(z) = z^3$.

(d) Yes. Take $v = -r^3 \cos 3\theta$ so that $u+iv = r^3(\sin 3\theta - i \cos 3\theta) = r^3(-i^2 \sin 3\theta - i \cos 3\theta) = r^3(-i)(\cos 3\theta + i \sin 3\theta) = -i(r^3 e^{3\theta i}) = -iz^3$.

(e) Yes. $v = r^2 \sin \theta$ so that $f(z) = z^2 + 4$.

(f) No.

(g) Yes. $v = y \cos 2x \cosh 2y - x \sin 2x \sin h2y$.